JOINT INFERENCE

BST228 Applied Bayesian Analysis

Speaker notes

No notes on this slide.

RECAP

- Inference of location μ given data $\mathbf{y}$ and precision τ.
- Inference of precision τ given data $\mathbf{y}$ and location μ.

Speaker notes

- We considered univariate inferences, assuming one of the parameters was known.
- The natural next step is to infer both parameters together.
- In the instrument manufacturer example, we may want to jointly estimate the concentration of a marker in a sample and the measurement error by running replicates. This means we do not need to rely on the reported precision of the instrument.

JOINT INFERENCE IN THEORY

$$p\left(\boldsymbol{\theta} \mid \mathbf{y}\right) = \frac{p\left(\mathbf{y} \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta}\right)}{p\left(\mathbf{y}\right)}$$

Speaker notes

- We often do not know any of the parameters of our model and need to infer them jointly.
- This includes hierarchical models, regression, non-parametric models, etc.
- Bayes theorem remains unchanged for multivariate inference, interpretation of prior, likelihood, and posterior stay the same.
- However, priors and posteriors are now multivariate distributions which require careful analysis.

JOINT INFERENCE IN PRACTICE

We have posterior

$$p(\boldsymbol{\theta} \mid \mathbf{y}) = p(\theta_1, \ldots, \theta_q \mid \mathbf{y})$$

for q parameters.

Speaker notes

- We need to handle high-distributions in `q` dimensions which can be both computationally and conceptually challenging.
- We often report summaries of the posterior, such as marginal distributions for each of he parameters (see next slide).

JOINT AND MARGINAL DISTRIBUTIONS

For a two-parameter posterior, by the law of total probability,

$$\begin{aligned} p\left(\theta_1 \mid \mathbf{y}\right) &= \int d\theta_2 \, p\left(\theta_1, \theta_2 \mid \mathbf{y}\right) \\ &= \int d\theta_2 \, p\left(\theta_1 \mid \theta_2, \mathbf{y}\right) p\left(\theta_2 \mid \mathbf{y}\right). \end{aligned}$$

The marginal posterior $p\left(\theta_1 \mid \mathbf{y}\right)$ is an average of the conditional posterior $p\left(\theta_1 \mid \theta_2, \mathbf{y}\right)$ weighted by the marginal posterior $p\left(\theta_2 \mid \mathbf{y}\right)$. Here, θ_2 is a nuisance variable that is not of primary concern. The integral is referred to as marginalization.

Speaker notes

No notes on this slide.

MARGINAL DISTRIBUTION EXAMPLE

The marginal posterior chemical concentration is

$$p\left(\mu \mid \mathbf{y}\right) = \int d\tau \, p\left(\mu, \tau \mid \mathbf{y}\right) .$$

Speaker notes

- The distribution of likely concentrations is what we are ultimately interested in.
- We thus marginalize with respect to the instrument precision, and the precision τ is a nuisance parameter.
- But before we can evaluate the marginal posterior, we need to obtain the joint posterior.

JOINT POSTERIOR FOR NORMAL DATA

The joint posterior is

$$\begin{aligned} p(\mu, \tau \mid \mathbf{y}) &\propto p(\mu, \tau)\, p(\mathbf{y} \mid \mu, \tau) \\ &\propto p(\mu, \tau) \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left(-\frac{\tau \sum_{i=1}^{n} (y_i - \mu)^2}{2}\right). \end{aligned}$$

We use a prior ansatz $\tau \sim \text{Gamma}(a_0, b_0)$ and $\mu \sim \text{Normal}\left(\nu_0, \frac{1}{\kappa_0 \tau}\right)$ such that

$$\begin{aligned} p(\mu, \tau \mid \mathbf{y}) &\propto \tau^{1/2} \exp\left(-\frac{\kappa_0 \tau}{2}(\mu - \nu_0)^2\right) \tau^{a_0 - 1} \exp(-b_0 \tau)\, \tau^{n/2} \exp\left(-\frac{\tau n}{2} \sum_{i=1}^{n} \frac{y_i^2 - 2\mu y_i + y_i^2}{n}\right) \\ &\propto \tau^{a_0 - 1 + \frac{n+1}{2}} \exp\left(-\tau \left[b_0 + \frac{\kappa_0}{2}(\mu^2 - 2\mu\nu_0 + \nu_0^2) + \frac{n}{2}(s - 2\mu\bar{y} + \mu^2)\right]\right), \end{aligned}$$

where $s = \sum_{i=1}^{n} \frac{y_i^2}{n}$ is the second moment of the sample. We note $a_n = a_0 + \frac{n}{2}$ and consider the term in brackets, which we call L.

Speaker notes

- Deriving the joint posterior is tedious but a worthwhile exercise.
- This derivation is the most fiddly algebra of the course, but I encourage you to verify the derivation in your own time.
- Using samplers to explore the posterior (see later lectures) allows us to side-step these derivations.
- Aside: Bayes had a bit of a revival starting in the late 90s because computational statistics became feasible.

JOINT POSTERIOR FOR NORMAL DATA

$$
\begin{aligned}
L &= b_0 + \frac{\kappa_0}{2}\left(\mu^2 - 2\mu\nu_0 + \nu_0^2\right) + \frac{n}{2}\left(s - 2\mu\bar{y} + \mu^2\right) \\
&= b_0 + \frac{1}{2}\left(\kappa_0\mu^2 - 2\kappa_0\mu\nu_0 + \kappa_0\nu_0^2 + ns - 2n\mu\bar{y} + n\mu^2\right) \\
&= b_0 + \frac{\kappa_0 + n}{2}\left(\mu^2 - 2\mu\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n} + \frac{\kappa_0\nu_0^2 + ns}{\kappa_0 + n}\right) \\
&= b_0 + \frac{\kappa_0 + n}{2}\left(\mu^2 - 2\mu\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n} + \left(\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}\right)^2 - \left(\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}\right)^2 + \frac{\kappa_0\nu_0^2 + ns}{\kappa_0 + n}\right) \\
&= b_0 + \frac{\kappa_0 + n}{2}\left(\mu - \frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}\right)^2 + \frac{\kappa_0 + n}{2}\left(\frac{\kappa_0\nu_0^2 + ns}{\kappa_0 + n} - \left(\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}\right)^2\right).
\end{aligned}
$$

We note $\kappa_n = \kappa_0 + n$ and $\nu_n = \frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}$. We further consider the first and last terms which is b_n.

Speaker notes

No notes on this slide.

JOINT POSTERIOR FOR NORMAL DATA

$$
\begin{aligned}
b_n &= b_0 + \frac{\kappa_0 + n}{2}\left(\frac{\kappa_0\nu_0^2 + ns}{\kappa_0 + n} - \left(\frac{\kappa_0\nu_0 + n\bar{y}}{\kappa_0 + n}\right)^2\right) \\
&= b_0 + \frac{1}{2}\left(\kappa_0\nu_0^2 + ns - \frac{\kappa_0^2\nu_0^2 + 2\kappa_0\nu_0 n\bar{y} + n^2\bar{y}^2}{\kappa_0 + n}\right) \\
&= b_0 + \frac{\left(\kappa_0\nu_0^2 + ns\right)\left(\kappa_0 + n\right) - \kappa_0^2\nu_0^2 - 2\kappa_0\nu_0 n\bar{y} - n^2\bar{y}^2}{2\left(\kappa_0 + n\right)} \\
&= b_0 + \frac{\kappa_0^2\nu_0^2 + n\kappa_0\nu_0^2 + \kappa_0 ns + n^2 s - \kappa_0^2\nu_0^2 - 2\kappa_0\nu_0 n\bar{y} - n^2\bar{y}^2}{2\left(\kappa_0 + n\right)} \\
&= b_0 + \frac{n}{2\left(\kappa_0 + n\right)}\left(\kappa_0\left(s - 2\nu_0\bar{y} + \nu_0^2\right) + n\left(s - \bar{y}^2\right)\right).
\end{aligned}
$$

Speaker notes

No notes on this slide.

UPDATE RULES

$$\kappa_n = \kappa_0 + n,$$

$$\nu_n = \frac{\kappa_0 \nu_0 + n\bar{y}}{\kappa_0 + n},$$

$$a_n = a_0 + \frac{n}{2},$$

$$b_n = b_0 + \frac{n}{2\left(\kappa_0 + n\right)}\left(\kappa_0\left(s - 2\nu_0\bar{y} + \nu_0^2\right) + n\mathbf{var}\,\mathbf{y}\right).$$

Speaker notes

- We derived four update rules for the posterior parameters, but the approach quickly becomes infeasible for more complex models.
- We have the joint posterior, and we can evaluate the marginal posterior distribution.
- In your own time, consider the limiting cases of large observation precision $\tau \to \infty$, large sample size $n \to \infty$, and large prior precision $\kappa_0 \to \infty$. Do the limiting cases agree with your intuition?

```
> # Define some random variables with correct support.
> n <- 7
> y <- rnorm(n)
> tau <- rgamma(1, 5, 5)
> mu <- rnorm(1)
> nu_0 <- rnorm(1)
> kappa_0 <- rgamma(1, 5, 5)
> a_0 <- rgamma(1, 5, 5)
> b_0 <- rgamma(1, 5, 5)
>
> reference <- dnorm(mu, mean = nu_0, sd = 1 / sqrt(kappa_0 * tau),
+   log = TRUE) + dgamma(tau, shape = a_0, rate = b_0, log = TRUE) +
+   sum(dnorm(y, mean = mu, sd = 1 / sqrt(tau), log = TRUE))
> print(paste("reference", reference))
[1] "reference -14.8945129997231"
>
> # Replace distributions by explicit values.
> test_value <- log(kappa_0 * tau / (2 * pi)) / 2 - kappa_0 * tau / 2 *
+   (mu - nu_0)^2 + a_0 * log(b_0) - lgamma(a_0) + (a_0 - 1) * log(tau) -
+   b_0 * tau + n * log(tau / (2 * pi)) / 2 - tau / 2 * sum((y - mu)^2)
>
> stopifnot(all.equal(test_value, reference))
>
> # Group terms and introduce a normalization constant to absorb terms.
> evaluate_log_norm <- function(n, kappa, a, b) {
+   return(
+     (log(kappa) - (n + 1) * log(2 * pi)) / 2 +
+     a * log(b) - lgamma(a)
+   )
+ }
```


Speaker notes

- Using R or another programming language can be a convenient way to verify algebraic manipulation by evaluating the manipulated expressions at some arbitrary values.

MARGINAL POSTERIOR FOR μ

Recall

$$p(\mu \mid \mathbf{y}, a_n, b_n) = \int d\tau \sqrt{\frac{\kappa_n \tau}{2\pi}} \exp\left(-\frac{\kappa_n \tau}{2}(\mu - \nu_n)^2\right) \frac{b_n^{a_n}}{\Gamma(a_n)} \tau^{a_n - 1} \exp(-b_n \tau)$$
$$\propto \int d\tau\, \tau^{a_n + 1/2 - 1} \exp\left(-\left(b_n + \frac{\kappa_n (\mu - \nu_n)^2}{2}\right)\tau\right),$$

where $\{\nu_n, \kappa_n, a_n, b_n\}$ are posterior parameters. The integrand is the kernel of a gamma distribution with effective parameters.

$$a' = a_n + \frac{1}{2}$$
$$b' = b_n + \frac{\kappa_n (\mu - \nu_n)^2}{2}.$$

Speaker notes

No notes on this slide.

MARGINAL POSTERIOR FOR μ

The integral thus evaluates to the inverse normalization constant $\Gamma(a')b'^{-a'}$, and

$$p\left(\mu \mid \mathbf{y}\right) \propto \left(1 + \frac{a_n \kappa_n \left(\mu - \nu_n\right)^2}{2a_n b_n}\right)^{-\frac{2a_n+1}{2}},$$

where we have absorbed a factor of b_n in the normalization constant and added a factor a_n to nominator and denominator. We compare the expression with the kernel of a non-centered, scaled Student's t-distribution

$$\left(1 + \frac{\kappa \left(\mu - \nu\right)^2}{q}\right)^{-\frac{q+1}{2}}$$

with q degrees of freedom, location ν, and precision κ.

Speaker notes

No notes on this slide.

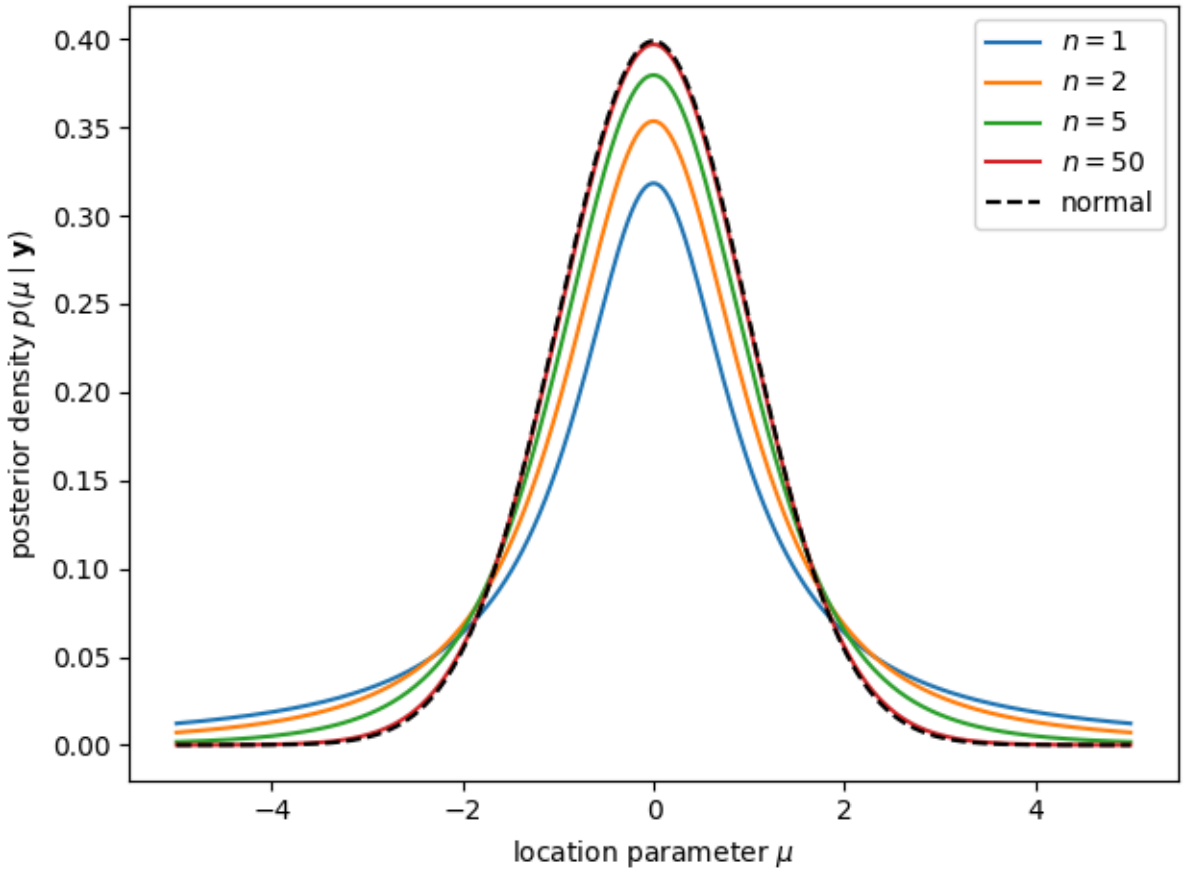
MARGINAL POSTERIOR FOR μ

Matching terms, we have a non-centered, scaled Student's t-distribution with $2a_n$ degrees of freedom. The marginal posterior is

$$\mu \mid \mathbf{y} \sim \text{StudentT}_{2a_n}\left(\nu_n, \frac{b_n}{\kappa_n a_n}\right).$$

Speaker notes

No notes on this slide.



Speaker notes

- The distribution, here shown for $\nu_n = 0$ has heaver tails than a normal distribution with the same parameters. However, even for relatively small sample size of $n = 50$, the Student's t-distribution closely approximates a normal distribution.
- This extra variance in the posterior for the chemical concentration is expected because we must also infer the observation precision given replicate measurements.

PAIRED EXERCISE

Consider again the example data $\mathbf{y} = (2.1, 2.5, 1.6, 1.7)$.

- Using the derived update rules, what are the posterior parameters?
- Draw posterior samples of μ? Can you think of two ways to obtain the samples?
- How do summary statistics of the posterior for μ and τ compare with inference assuming one known parameter?

Speaker notes

No notes on this slide.

```
> # Declare data and known noise level.
> y <- c(2.1, 2.5, 1.6, 1.7)
> n <- length(y)
> # Define hyperparameters.
> nu_0 <- 0
> kappa_0 <- 1e-4
> a_0 <- 1e-3
> b_0 <- 1e-3
> # Update parameters and sample.
> nu_n <- (nu_0 * kappa_0 + n * mean(y)) / (kappa_0 + n)
> kappa_n <- kappa_0 + n
> a_n <- a_0 + n / 2
> b_n <- b_0 + n / 2 * (var(y) * n / (n - 1) + kappa_0 / (kappa_0 + n)
+                       * (mean(y) - nu_0) ** 2)
> tau_samples <- rgamma(1000, a_n, b_n)
> sigma_samples <- 1 / sqrt(tau_samples)
> mu_samples <- rnorm(1000, nu_n, 1 / sqrt(kappa_n * tau_samples))
> c(mean(mu_samples), sd(mu_samples),
+   mean(sigma_samples), sd(sigma_samples))
[1] 1.9708200 0.3235266 0.6033182 0.3464679
>
```

Speaker notes

- Lines #2-3 declare data and #5-8 declare hyperparameters.
- #10-14 evaluate the posterior parameters.
- #15-17 sample from the posterior in two steps. Alternatively, we could have directly sampled from a Student's t-distribution.